

Valuation Adjustments with an Affine-Diffusion-based Interest Rate Smile Rabobank & Utrecht University, the Netherlands

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Outline

Goal: incorporate smiles in Valuation Adjustments (xVAs).

Steps:

- 1 Introduction.
- 2 Our contribution.
- 3 SDE with state-dependent drift / diffusion.
- 4 Randomized Affine Diffusion (RAnD).
- 5 Calibration, simulation and pricing.
- 6 Conclusions.

Introduction

- 1 Background on xVAs:
 - a Economic value = risk-neutral value – xVA.
 - b Valuation Adjustments (xVAs), e.g., CVA, DVA, FVA, MVA, KVA.
 - c Computational challenges.
- 2 Focus on xVAs for IR derivatives.
- 3 Common xVA modeling setup in a Monte Carlo framework:
 - a Use one-factor short-rate model in Affine Diffusion class.
 - b Analytic tractability motivates use for xVA purposes.
 - c Example: Hull-White one-factor model (HW1F).

HW1F model

- ❶ Impossible to fit to the whole market volatility surface (expiry \times tenor \times strike).
- ❷ Time-dependent piece-wise constant volatility parameter used to calibrate the model to a strip of ATM co-terminal swaptions.
- ❸ Forward rate under HW1F is shifted-lognormal: there is skew but it cannot be controlled.
- ❹ The model does not generate volatility smile.
- ❺ HW1F dynamics in the G1++ form:

$$r(t) = x(t) + b(t), \quad dx(t) = -a_x x(t)dt + \sigma_x(t)dW(t).$$

Smile and skew: the market

- 1 Volatility smile on the short end.
- 2 Transforms into skew over time.

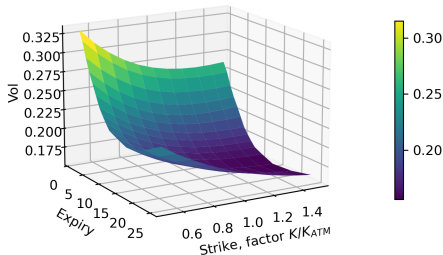
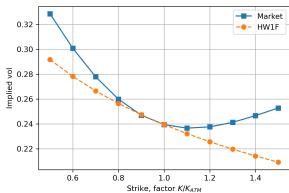
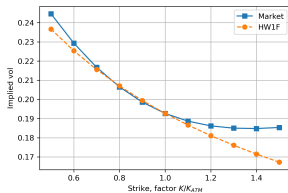


Figure: USD swaption volatility surface with 10Y tenor, market data from 28/09/2022. The volatilities are shifted Black volatilities. The strike is given as a factor times the ATM strike K_{ATM} , e.g., 1.2 means a strike of $1.2 \cdot K_{ATM}$.

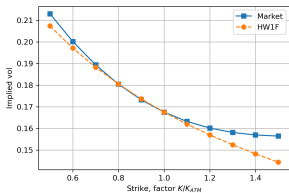
Smile and skew: the market vs HW1F



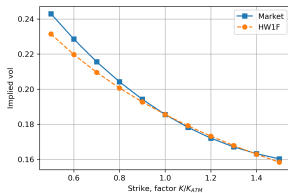
(a) 1Y expiry, 29Y tenor.



(b) 5Y expiry, 25Y tenor.



(c) 10Y expiry, 20Y tenor.



(d) 25Y expiry, 5Y tenor.

Figure: USD 30Y co-terminal swaption volatility strips (02/12/2022).

Smile and skew: xVA

- 1 Smile and skew typically absent from xVA calculations.
- 2 Challenge: find a model that captures smile and skew, but also allows for efficient calibration and pricing.
- 3 Smile and skew can be relevant for xVA:
 - a Obvious case: derivatives that take into account smile.
 - b Also for linear derivatives: legacy trades that are off-market and not primarily driven by ATM vols.
 - c Larger effect expected on PFE as this is a tail metric.
- 4 Examples in literature:
 - a Andreasen used a four-factor Cheyette model with local and stochastic volatility [1].
 - b Quadratic Gaussian models (quadratic form for the short rate) also allow smile control [3, Section 16.3.2].

Our contribution

- 1 Find SDE with state-dependent drift / diffusion that is consistent with the convex combination of N different HW1F models, where one model parameter is varied.
- 2 This model allows to capture market smile and skew.
- 3 Profit from the analytic tractability of Affine Diffusion dynamics.
- 4 The model allows for fast and semi-analytic swaption calibration.
- 5 Monte Carlo pricing using regression methods.
- 6 Use the idea of the RAnD method to parameterize the model: one additional degree of freedom for HW1F.

SDE with state-dependent drift / diffusion

- 1 General dynamics for $r(t)$ for which we try to find the (potentially) state-dependent drift and diffusion:

$$dr(t) = \mu_r(t, r(t))dt + \eta_r(t, r(t))dW(t). \quad (1)$$

- 2 We want to find $\mu_r(t, r(t))$ and $\eta_r(t, r(t))$ s.t. $\forall t$ the density is consistent with the convex combination of N densities of analytically tractable models $r_n(t)$:

$$f_{r(t)}(y) = \sum_{n=1}^N \omega_n f_{r_n(t)}(y), \quad (2)$$

where

$$dr_n(t) = \mu_{r_n}(t, r_n(t))dt + \eta_{r_n}(t, r_n(t))dW(t). \quad (3)$$

- 3 $\sum_{n=1}^N \omega_n = 1$ and $\omega_n > 0 \forall n$.
- 4 We derive $\mu_r(t, r(t))$ and $\eta_r(t, r(t))$ using the Fokker-Planck eq.

Fokker-Planck: applied to our case

We write down the FP equation for both $r(t)$ and $r_n(t)$. Using

$$f_{r(t)}(y) = \sum_{n=1}^N \omega_n f_{r_n(t)}(y), \quad (4)$$

and linearity of the derivative operator we obtain:

$$dr(t) = \mu_r(t, r(t))dt + \eta_r(t, r(t))dW(t), \quad (5)$$

$$\mu_r(t, y) = \sum_{n=1}^N \mu_{r_n}(t, y) \Lambda_n(t, y), \quad (6)$$

$$\eta_r^2(t, y) = \sum_{n=1}^N \eta_{r_n}^2(t, y) \Lambda_n(t, y), \quad (7)$$

$$\Lambda_n(t, y) = \frac{\omega_n f_{r_n(t)}(y)}{\sum_{i=1}^N \omega_i f_{r_i(t)}(y)}. \quad (8)$$

The $r_n(t)$ dynamics

We work with the HW1F model in the G1++ formulation, where each $r_n(t)$ has a different mean-reversion θ_n :

$$r_n(t) = x_n(t) + b_n(t), \quad (9)$$

$$dx_n(t) = -\theta_n x_n(t)dt + \sigma_x dW(t), \quad (10)$$

$$b_n(t) = f^M(0, t) - x_n(0)e^{-\theta_n t} + \frac{1}{2}\sigma_x^2 B_n^2(0, T), \quad (11)$$

$$B_n(s, t) = \frac{1}{\theta_n} \left(1 - e^{-\theta_n(t-s)}\right). \quad (12)$$

- Constant volatility σ_x for ease of notation, in reality piece-wise constant $\sigma_x(t)$ is used.
- $r_n(t) \sim \mathcal{N}(\mathbb{E}_s[x_n(t)] + b_n(t), \text{Var}_s(x_n(t)))$ conditional on \mathcal{F}_s .
- So $f_{r_n(t)}(y)$ is a normal pdf.

The $r_n(t)$ dynamics

Writing these dynamics in the desired form

$$dr_n(t) = \mu_{r_n}(t, r_n(t))dt + \eta_{r_n}(t, r_n(t))dW(t) \quad (13)$$

yields

$$\mu_{r_n}(t, r_n(t)) = \frac{d f^M(0, t)}{d t} + \theta_n f^M(0, t) - \theta_n r_n(t) + \text{Var}_0(r_n(t)), \quad (14)$$

$$\eta_{r_n}(t, r_n(t)) = \sigma_x. \quad (15)$$

The $r(t)$ dynamics

Using these results, we have that

$$dr(t) = \mu_r(t, r(t))dt + \eta_r(t, r(t))dW(t), \quad (16)$$

$$\mu_r(t, r(t)) = \sum_{n=1}^N \left[\frac{df^M(0, t)}{dt} + \theta_n f^M(0, t) - \theta_n r(t) + \text{Var}_0(r_n(t)) \right] \cdot \Lambda_n(t, r(t)), \quad (17)$$

$$\eta_r(t, r(t)) = \sqrt{\sum_{n=1}^N \sigma_x^2 \cdot \Lambda_n(t, r(t))} = \sigma_x, \quad (18)$$

as $\sum_{n=1}^N \Lambda_n(t, y) = 1 \forall y$.

This means that the diffusion component $\eta_r(t, r(t))$ is unchanged, whereas the drift $\mu_r(t, r(t))$ is **state-dependent**.

Convex combinations of dynamics

We derived the dynamics $X(t)$ that corresponds to the convex combination of N different models $X_n(t)$:

- 1 The resulting model is then driven by $f_{X(t)}(y)$:

$$f_{X(t)}(y) = \sum_{n=1}^N \omega_n f_{X_n(t)}(y). \quad (19)$$

- 2 A similar result holds for the valuation of a derivative $V_X(t)$:

$$V_X(t) = \sum_{n=1}^N \omega_n V_{X_n(t)}. \quad (20)$$

- 3 Eq. (20) obtained for call option on equity when imposing (19) under the T -forward measure.
- 4 Eq. (20) holds for non-path-dependent derivatives only.
- 5 For more complex derivatives, derive state-dependent (local-vol type) dynamics as before.

Randomized Affine Diffusion

Randomized Affine Diffusion (RAnD) method [4, 5]:

- 1 Take an Affine Diffusion (AD) model.
- 2 Pick model parameter ϑ to randomize.
- 3 The random variable ϑ is defined on domain $D_\vartheta := [a, b]$ with PDF $f_\vartheta(x)$ and CDF $F_\vartheta(x)$, and realization θ , $\vartheta(\omega) = \theta$, such that the moments are finite.
- 4 For valuation, we use Gauss-quadrature weights $\{\omega_n, \theta_n\}_{n=1}^N$ where the nodes θ_n are based on $F_\vartheta(x)$, see [5, Appendix A.2]. Then, for the valuation:

$$V(t, r(t; \vartheta)) = \int_{[a,b]} V(t, r(t; \theta)) dF_\vartheta(\theta) \approx \sum_{n=1}^N \omega_n V(t, r(t; \theta_n)).$$

RAnD for model parametrization

- 1 Use the idea of the RAnD method to reduce dimensionality of our model parameters.
- 2 We do not suffer from the quadrature error.
- 3 We work with the HW1F dynamics.
- 4 We choose $\vartheta = a_x$, i.e., the mean-reversion parameter.
- 5 Impose $\mathcal{N}(\mu_{\vartheta}, \sigma_{\vartheta}^2)$ as randomizer (constant over time).
- 6 $N = 5$ suitable when ϑ follows a normal (or uniform) distribution.
- 7 Key advantage: one additional degree of freedom w.r.t. HW1F.

Calibration of the $r(t)$ dynamics

① Calibration of the $r_n(t)$ HW1F dynamics in the usual way.

② Mean-reversion parameterized as $a_x \sim \mathcal{N}(\mu_\vartheta, \sigma_\vartheta^2)$.

For each choice of μ_ϑ and σ_ϑ^2 :

a Compute collocation points (Gauss-quad weights) $\{\omega_n, \theta_n\}_{n=1}^N$.

b Initialize N HW1F models with mean-reversion parameter $a_x = \theta_n$.

③ Use fast valuation

$$V(t, r(t)) = \sum_{n=1}^N \omega_n V(t, r_n(t)).$$

④ Calibrate the parametrization of the mean-reversion $a_x \sim \mathcal{N}(\mu_\vartheta, \sigma_\vartheta^2)$ according to the desired strategy:

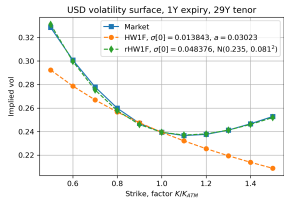
a Fit the initial coterminal smile.

b Fit all ATM points of the vol surface.

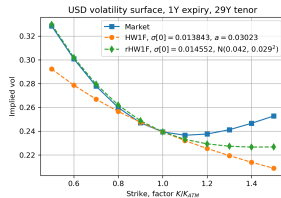
c Fit all coterminal smiles.

⑤ Bootstrap calibration of piece-wise constant model volatility to get a good ATM fit to the coterminal swaption strip.

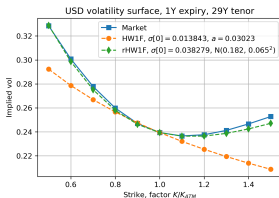
Calibration results



(a) Fit initial coterminal smile.



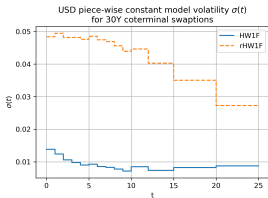
(b) Fit all ATM points.



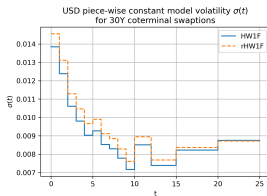
(c) Fit all coterminal smiles.

Figure: Initial coterminal smile. USD market data from 02/12/2022.

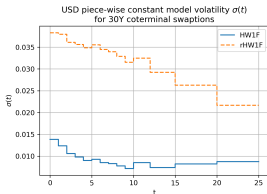
Calibration results



(a) Fit initial coterminal smile.



(b) Fit all ATM points.

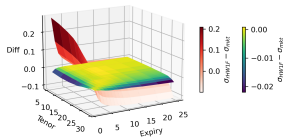


(c) Fit all coterminal smiles.

Figure: Calibrated model volatilities. USD market data from 02/12/2022.

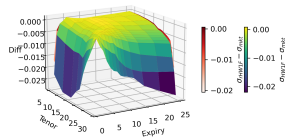
Calibration results

MSE Impvol surf: HW1F = 1.81e-04 & rHW1F = 4.44e-03
MSE Impvol ATM: HW1F = 5.81e-05 & rHW1F = 4.25e-03
MSE Impvol smile: HW1F = 5.07e-04 & rHW1F = 1.92e-06



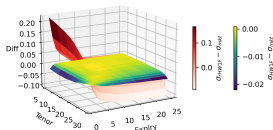
(a) Fit initial coterminal smile.

MSE Impvol surf: HW1F = 1.81e-04 & rHW1F = 1.12e-04
MSE Impvol ATM: HW1F = 5.81e-05 & rHW1F = 4.50e-05
MSE Impvol smile: HW1F = 5.07e-04 & rHW1F = 1.24e-04



(b) Fit all ATM points.

MSE Impvol ATM: HW1F = 5.81e-05 & rHW1F = 2.83e-03
MSE Impvol init smile: HW1F = 5.07e-04 & rHW1F = 6.63e-06
MSE Impvol cot smiles: HW1F = 8.15e-05 & rHW1F = 2.55e-06



(c) Fit all coterminal smiles.

Figure: Difference in ATM implied vols. USD market data from 02/12/2022.

Simulation of the $r(t)$ dynamics

Back to our dynamics:

$$dr(t) = \mu_r(t, r(t))dt + \eta_r(t, r(t))dW(t), \quad (21)$$

$$\mu_r(t, r(t)) = \sum_{n=1}^N \left[\frac{df^M(0, t)}{dt} + \theta_n f^M(0, t) - \theta_n r(t) + \text{Var}_0(r_n(t)) \right] \cdot \Lambda_n(t, r(t)), \quad (22)$$

$$\eta_r(t, r(t)) = \sqrt{\sum_{n=1}^N \sigma_x^2 \cdot \Lambda_n(t, r(t))} = \sigma_x, \quad (23)$$

as $\sum_{n=1}^N \Lambda_n(t, y) = 1 \forall y$.

This means that the diffusion component $\eta_r(t, r(t))$ is unchanged, whereas the drift $\mu_r(t, r(t))$ is **state-dependent**.

Simulation of the $r(t)$ dynamics

- 1 Euler-Maruyama discretization always works:

$$r(t_{i+1}) = r(t_i) + \mu_r(t_i, r(t_i))\Delta t + \eta_r(t, r(t_i))\sqrt{\Delta t}Z, \quad (24)$$

where $Z \sim \mathcal{N}(0, 1)$.

- 2 Ideally we make large time steps. Hence, we integrate $dr(t)$ to obtain an expression for $r(t)$ conditional on $r(s)$ for $s < t$, i.e.,

$$r(t) = r(s) + \int_s^t \mu_r(u, r(u))du + \int_s^t \eta_r(u, r(u))dW(u). \quad (25)$$

- 3 The integrated drift is difficult to compute:

$$\begin{aligned} \int_s^t \mu_r(u, r(u))du &= f^M(0, t) - f^M(0, s) \\ &+ \int_s^t \sum_{n=1}^N \left[\theta_n f^M(0, u) - \theta_n r(u) + \mathbb{V}ar_0(r_n(u)) \right] \Lambda_n(u, r(u))du. \end{aligned}$$

- 4 Alternatively: machine learning, e.g., Seven-League scheme [6].

Pricing under the $r(t)$ dynamics

- 1 Valuation as convex combination of underlying prices only for Europeans.
- 2 In general, we can use Monte Carlo with regression:
 - a Regression to avoid nested simulation.
 - b For example, we simulate r from t_0 to t and at this point we want to compute $P(t, T) = \mathbb{E}_t \left[e^{-\int_t^T r(s) ds} \right]$.
 - c For each T where we need $P(t, T)$ in pricing, it is regressed on $r(t)$.
 - d For example, an n -th order polynomial can be used as regression function, or something of exponential form.
- 3 These regression-based methods lend themselves naturally for xVA calculations, also known as American Monte Carlo.

Pricing a swaption under the $r(t)$ dynamics

- 1 Swaption with 10k notional, 5y expiry, on a 5y payer swap with annual payments.
- 2 Use 10^5 MC paths (antithetic variates turned on) and 100 simulation dates per year.
- 3 Polynomial regression of degree 4.

	Value	Imp.vol
HW1F: analytic	328.63814	0.22186
Convex comb: analytic	580.31577	0.40080
Convex comb: MC regressed ZCB	582.41497	0.40235
RAnD dynamics: MC regressed ZCB	581.20828	0.40146
Abs diff	1.20669	8.92e-04
Rel diff	2.08e-03	

Table: Results for all coterminal smiles calibration. Absolute and relative differences are between convex combination and RAnD dynamics values using the MC with regressed ZCB. RAnD 95% conf.int.: (578.96, 583.46).

Conclusions

- 1 Find SDE with state-dependent drift / diffusion that is consistent with the convex combination of N different HW1F models, where one model parameter is varied.
- 2 This model allows to capture market smile and skew.
- 3 Profit from the analytic tractability of Affine Diffusion dynamics.
- 4 The model allows for fast and semi-analytic swaption calibration.
- 5 Monte Carlo pricing using regression methods.
- 6 Use the idea of the RAnD method to parameterize the model: one additional degree of freedom for HW1F.

Valuation Adjustments with an Affine-Diffusion-based Interest Rate Smile

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